

# On the deconfining limit in (2+1)-dimensional Yang-Mills theory

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## Abstract

We consider (2+1)-dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  in the framework of a Hamiltonian approach developed by Karabali, Kim and Nair. The deconfining limit in the theory can be discussed in terms of one of the  $S^1$  radii of the torus ( $S^1 \times S^1$ ), while the other radius goes to infinity. We find that the limit agrees with the previously known result for a dynamical propagator mass of a gluon. We also make comparisons with numerical data.

# 1 Introduction

The Hamiltonian approach to (2+1)-dimensional Yang-Mills theory has been known over a decade as a novel framework for non-perturbative analysis of the theory [1]. Technical elaborations might be necessary in the formulation of the Hamiltonian approach, in particular, with regard to regularization processes [2], but the upshot of the Hamiltonian formulation is quite simple; namely, it gives rise to (a) an interpretation of an origin of the mass gap and (b) an analytic calculation of the string tension for  $SU(N)$  gauge groups [2, 3]. These results, obtained by Karabali, Kim and Nair (KKN), are remarkable not only in consequence of a (conformal) field theoretic framework but also in comparison with lattice simulations of the string tension [4]. Recent developments relevant to the Hamiltonian approach can be found in [5, 6, 7, 8, 9]. There are also various other analytic approaches to the theory; for recent progress, see, for example, [10, 11, 12].

In the present paper, we follow the Hamiltonian approach to consider the deconfining limit in (2+1)-dimensional Yang-Mills theory. In the Hamiltonian approach, confining properties of the theory can be shown by the following steps [3]:

1. matrix parametrization of gauge fields;
2. calculation of a gauge-invariant measure on the configuration space;
3. evaluation of a vacuum-state wave function  $\Psi_0$  and its inner product in terms of gauge-invariant variables;
4. calculation of the vacuum expectation value of the Wilson loop operator  $\langle \Psi_0 | W(C) | \Psi_0 \rangle$ ; and
5. reading off the area law or the positive string tension from  $\langle \Psi_0 | W(C) | \Psi_0 \rangle$ .

The goal of this paper is to rephrase each of the above steps for the theory on  $S^1 \times S^1 \times \mathbf{R}$  such that we can discuss the limit of vanishing string tension in terms of one of the  $S^1$  radii which may correspond to a parameter of finite temperature. For such a parameter we will use  $\text{Im}\tau$ , where  $\tau$  is the modular parameter of a torus ( $S^1 \times S^1$ ) in the three-dimensional space  $S^1 \times S^1 \times \mathbf{R}$  of our interest. A radius corresponding to the other  $S^1$  is taken to be large (or equivalently the corresponding winding number is taken to be large, with the radius being finite) so that we can identify the theory on  $S^1 \times S^1 \times \mathbf{R}$  as a planar theory at a finite temperature in such a limit. This implies that one of the  $S^1$  directions corresponds to the time coordinate.

In the KKN Hamiltonian approach, one takes the temporal gauge  $A_0 = 0$  and analyses are made entirely on complexified spatial dimensions by use of conformal properties. In the present paper, we however take a different gauge, say  $A_x = 0$ , to make an analysis of gauge potentials on torus which include a time component. Such an analysis may contain subtlety in discussion of physical dynamics but for argument of static properties it may still be useful since the theory of interest is relevant to the one with imaginary time or the Euclidean metric. In this paper, leaving that subtlety aside, we focus rather on construction

of the vacuum wave function  $\Psi_0$  in pure Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  (with one of  $S^1$  directions corresponding to an imaginary time coordinate), following the framework of the KKN Hamiltonian approach. We shall not deal with a more involved regularization program, either. Note that (2+1)-dimensional Yang-Mills theory on torus in spatial dimensions has been studied before in a different context [13].

Apart from what have been mentioned, our main motivation to consider the deconfining limit is currently available lattice data on the deconfinement phase transition in (2+1)-dimensional Yang-Mills theory [14, 15, 16, 17, 18, 19]. In the Hamiltonian approach, the string tension is obtained for a continuum strong coupling region where  $e^2/p \gg 1$  is realized, with  $e$  and  $p$  being the coupling constant and a typical momentum scale, respectively. So we shall limit our analysis in this region and do not discuss the nature of deconfinement phase transition. What we aim at is, however, to obtain a critical temperature of deconfinement transition which can be compared with the numerical data.

Study of (2+1)-dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  is therefore physically well-motivated. In order to execute the study in the Hamiltonian approach, however, there is a key mathematical concept to be reminded of, that is, physical states of the planar Yang-Mills theory in the Hamiltonian approach can be described in terms of holomorphic wave functionals of Chern-Simons theory. In the next section, we briefly review this relation. Once we understand how this relation arises, consideration of the toric theory would be clearer by use of the so-called Narashimhan-Seshadri theorem [20], which we also mention in the next section.

In section 3, following the references [21, 22, 23], we present a matrix-parametrization of gauge potentials such that it is incorporated with the zero modes of torus. This section covers the first two steps of the above list. In section 4, we deal with the third step. Technically speaking, the objective of this section is to find holomorphic functionals of Chern-Simons theory on torus in a context of geometric/holomorphic quantization [22, 23]. In section 5, utilizing the results of section 4, we consider the last two steps of the above list. We obtain an expression for a deconfinement temperature and make comparisons with lattice data. In the last section, we present brief concluding remarks.

## 2 Review of the KKN Hamiltonian approach

In the Hamiltonian approach, the gauge potentials  $A_i$  ( $i = 1, 2, 3$ ) are parametrized by the elements of  $SL(N, \mathbf{C})$ . The gauge group we consider is  $G = SU(N)$ ;  $A_i$  can be written as  $A_i = -it^a A_i^a$  where  $t^a$ 's are the elements of  $SU(N)$  represented by  $(N \times N)$ -matrices, satisfying  $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$  and  $[t^a, t^b] = if^{abc} t^c$ . Note that under the temporal gauge  $A_0 = 0$  the gauge potentials are described by  $A_z = \frac{1}{2}(A_1 + iA_2)$ ,  $A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2)$  where  $z = x_1 - ix_2$ ,  $\bar{z} = x_1 + ix_2$  are a complex combination of the spatial coordinates  $(x_1, x_2)$ . Matrix parametrization of the gauge potentials is given by

$$\begin{aligned} A_z &= -\partial_z M M^{-1} \\ A_{\bar{z}} &= M^{\dagger-1} \partial_{\bar{z}} M^{\dagger} \end{aligned} \tag{1}$$

where  $M(z, \bar{z}), M^\dagger(z, \bar{z})$  are the elements of  $SL(N, \mathbf{C})$ . Gauge transformations of  $A_z, A_{\bar{z}}$  can be realized by  $M \rightarrow gM, M^\dagger \rightarrow M^\dagger g^{-1}$  with  $g \in SU(N)$ . A gauge invariant matrix variable is given by  $H = M^\dagger M$ . The parametrization (1) corresponds to step 1 of the list in the introduction.

Let  $\mathcal{A}$  denote the set of all gauge potentials  $A_i^a$ . The gauge-invariant configuration space is then given by

$$\mathcal{C} = \mathcal{A}/\mathcal{G}_* \quad (2)$$

where

$$\mathcal{G}_* = \{\text{set of all } g(\vec{x}) : \mathbf{R}^2 \rightarrow SU(N), \text{ with } g \rightarrow 1 \text{ as } |\vec{x}| \rightarrow \infty\}. \quad (3)$$

The calculation of the gauge-invariant measure in step 2 leads to the following result, up to an irrelevant constant factor [1]:

$$d\mu(\mathcal{C}) = d\mu(H) e^{2c_A S_{WZW}(H)} \quad (4)$$

where  $c_A$  denotes the quadratic Casimir of  $G$  for the adjoint representation,  $c_A \delta^{ab} = f^{amn} f^{bm n}$ . For  $G = SU(N)$ , this is equal to  $N$ .  $S_{WZW}(H)$  is the action for a  $G^{\mathbf{C}}/G$  Wess-Zumino-Witten (WZW) model where  $G^{\mathbf{C}} = SL(N, \mathbf{C})$  is the complexification of  $G = SU(N)$ . Explicitly, the action is given by

$$S_{WZW}(H) = \frac{1}{2\pi} \int d^2 z \text{Tr}(\partial_z H \partial_{\bar{z}} H^{-1}) + \frac{i}{12\pi} \int d^3 x \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H) \quad (5)$$

where  $d^2 z$  (or  $dx^2$ ) is a real two-dimensional volume element which is equivalent to  $dz d\bar{z}/2i$ . For description of a coset model in connection with gauged WZW models, one may refer to [21, 24, 25, 26]. The inner product of physical states in general, not only for vacuum states, can be evaluated as correlators of the  $G^{\mathbf{C}}/G$ -WZW model;

$$\langle 1|2 \rangle = \int d\mu(H) e^{2c_A S_{WZW}} \Psi_1^*(H) \Psi_2(H). \quad (6)$$

where  $\Psi^*(H)$  indicates the complex conjugate of a wave functional  $\Psi(H)$ . The wave functionals for the *vacuum state* can be given by

$$\Psi_0(H) = 1. \quad (7)$$

Equation (6) provides an essential setup for the KKN Hamiltonian approach.

It is known that current correlators of a WZW model can be generated by holomorphic wave functionals of Chern-Simons theory [22, 23].<sup>1</sup> The wave functional  $\Psi(H)$  can then be interpreted as a functional which arises from a holomorphic wave functional  $\psi[A_{\bar{z}}]$  of Chern-Simons theory. In a context of geometric/holomorphic quantization of Chern-Simons theory [22, 23, 28], a polarization condition is imposed on the functional, *i.e.*,

$$\Psi[A_{\bar{z}}] = e^{-\frac{\kappa}{2}} \psi[A_{\bar{z}}]. \quad (8)$$

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<sup>1</sup>It is well known by the study of knots in terms of Chern-Simons theory [27] that the conformal blocks of current algebra on  $\Sigma$  (or the current correlators of a WZW model on  $\Sigma$ ) correspond to the sections of holomorphic line bundle over  $\mathcal{M}$  (or the generating functional of Chern-Simons theory on  $\mathcal{M}_3$ ), where  $\Sigma$  is a two-dimensional compact space,  $\mathcal{M}$  is a moduli space of flat connections on a  $G$ -bundle over  $\Sigma$  and  $\mathcal{M}_3$  is a three-dimensional closed manifold with its boundary being  $\partial\mathcal{M}_3 = \Sigma$ .

Here  $K$  is a Kähler potential associated with the phase space of Chern-Simons theory in the  $A_0 = 0$  gauge;

$$K = \frac{k}{2\pi} \int_{\Sigma} A_{\bar{z}}^a A_z^a = -\frac{k}{\pi} \int_{\Sigma} \text{Tr}(A_{\bar{z}} A_z) \quad (9)$$

where  $\Sigma$  indicates Riemann surface and  $k$  is the level number of Chern-Simons theory. The integral over  $\Sigma$  is taken for  $dz^2 = dzd\bar{z}/2i$ . In terms of the parametrization (1),  $\psi[A_{\bar{z}}]$  can be written as  $\psi[M^{\dagger}]$ . The flatness of the gauge potentials, which is required as the equation of motion for  $A_0$  (or the Gauss law constraint) of Chern-Simons theory, must be satisfied on the holomorphic wave functional  $\psi[A_{\bar{z}}]$ , *i.e.*,  $F_{z\bar{z}}\psi[A_{\bar{z}}] = 0$  where  $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$ . This leads to  $\psi[M^{\dagger}] = e^{kS_{WZW}(M^{\dagger})}$ . The inner product of the gauge-invariant physical states is then given by [23]

$$\begin{aligned} \langle 1|2\rangle_{CS} &= \int d\mu(\mathcal{C}) e^{-K} \psi_1^* \psi_2 \\ &= \int d\mu(H) e^{(2c_A + k)S_{WZW}(H)} \end{aligned} \quad (10)$$

where we use (4) and  $\psi_1^* = \psi_1[M] = e^{kS_{WZW}(M)}$  together with the Polyakov-Wiegmann identity [29]

$$\begin{aligned} S_{WZW}(H) = S_{WZW}(M^{\dagger}M) &= S_{WZW}(M^{\dagger}) + S_{WZW}(M) - \frac{1}{\pi} \int_{\Sigma} \text{Tr}(M^{\dagger-1} \partial_{\bar{z}} M^{\dagger} \partial_z M M^{-1}) \\ &= S_{WZW}(M^{\dagger}) + S_{WZW}(M) + \frac{1}{\pi} \int_{\Sigma} \text{Tr}(A_{\bar{z}} A_z). \end{aligned} \quad (11)$$

Comparing (6) and (10), we find that the inner product of the vacuum wave functionals on the planar Yang-Mills theory can be obtained by the inner product (10) in the limit of  $k \rightarrow 0$ . The theory defined by the correlator (10) with positive  $k$  is known as Yang-Mills-Chern-Simons theory [30, 31, 32].

It is interesting that the physical states of (2+1)-dimensional Yang-Mills theory can be obtained in terms of the holomorphic wave functionals of Chern-Simons theory. A simple explanation of this relation is that, as shown in the first reference of [1], under the  $A_0 = 0$  gauge commutation rules among the gauge potentials  $A_i$ 's and the electric fields  $E_i$ 's (which are canonical momenta of  $A_i$ 's) can be interpreted as two copies of the Chern-Simons commutation rules among  $A_z$ 's and  $A_{\bar{z}}$ 's in the same gauge.

For physical states other than the vacuum, the holomorphic wave functional  $\psi[A_{\bar{z}}]$  of Chern-Simons theory may be expressed as  $\psi[M^{\dagger}] = e^{kS_{WZW}(M^{\dagger})} F[M^{\dagger}]$  in general where  $F[M^{\dagger}]$  is a matrix function of  $M^{\dagger}$ . Thus  $\psi[M^{\dagger}]$  may not lead to the gauge invariant functional  $\Psi(H)$  in (6). However, as shown in [8], one can in fact take a suitable gauge choice such that  $F[M^{\dagger}]$  depends on the current of the hermitian  $SL(N, \mathbf{C})/SU(N)$ -WZW model,  $J^a = \frac{c_A}{\pi} (\partial_z H H^{-1})^a$ . In terms of this gauge-invariant current, the flatness of the gauge potential corresponds to an equation of motion of the hermitian WZW model,  $\partial_{\bar{z}} J^a = 0$ . In the Hamiltonian approach, this indicates the vanishing of magnetic fields which act on  $\psi[A_{\bar{z}}]$ . Note that magnetic fields do not necessarily vanish when acted on  $\Psi[A_{\bar{z}}]$ . In the present paper, effects of magnetic fields will not be discussed in constructing the vacuum-state wave functional.

Now let us return to the toric theory. It is known that there exist flat connections (or non-trivial gauge potentials that lead to vanishing curvature) for any compact two-dimensional spaces  $\Sigma$  which have complex structure.<sup>2</sup> The relation between the current correlators of a WZW model and the holomorphic wave functions of Chern-Simons theory, given by the expression of (6) or (10), therefore holds for any Riemann surfaces  $\Sigma$  including a torus. For the study of WZW models on Riemann surfaces and on torus in particular, see [21, 33, 34]. A main purpose of the present paper from a mathematical perspective is to clarify the structure of the inner product of such holomorphic wave functionals when the Riemann surface is given by a torus.

As seen in the next section, one can in fact incorporate zero modes of torus into a matrix-parametrization of gauge potentials, analogous to the form in (1). What we need to do is therefore to obtain a toric version of (6) or (10) by use of such a matrix parametrization. A main claim we like to make in this paper is that, in the toric case, the level number appeared in (10) no longer vanishes since it is now incorporated with nontrivial zero-mode dynamics. If the toric level number, say  $\tilde{k}$ , vanishes, there will be no topological differences from the planar case. We discuss contributions of zero modes later in section 4.

### 3 Matrix parametrization of gauge potentials on torus

Torus can be described in terms of two real coordinates  $\xi_1, \xi_2$  with periodicity of  $\xi_i \rightarrow \xi_i + n$  ( $i = 1, 2$ ) where  $n$  is any integer. Complex coordinates of torus can be parametrized as  $z = \xi_1 + \tau \xi_2$  where  $\tau = \text{Re}\tau + i\text{Im}\tau$  is the modular parameter of the torus. There are two noncontractible cycles on torus, conventionally labeled as  $\alpha$  and  $\beta$  cycles. By use of these a holomorphic one-form of the torus,  $\omega = \omega(z)dz$ , can be defined as

$$\int_{\alpha} \omega = 1, \quad \int_{\beta} \omega = \tau \quad (12)$$

where the normalization of  $\omega$  is given by

$$\int dz d\bar{z} \bar{\omega} \wedge \omega = i2 \text{Im}\tau. \quad (13)$$

Equivalently, this can be written as  $\int_{\Sigma} \bar{\omega} \wedge \omega = \text{Im}\tau$  with  $\Sigma = S^1 \times S^1$ . Note that  $\omega$  is a zero mode of  $\partial_{\bar{z}}$ . In construction of matrix parametrization of gauge potentials on torus, we therefore need to take  $\omega$  and  $\bar{\omega}$  into account.

We shall denote  $a$  as a complex physical variable of zero modes for the moment. We may regard  $a$  as an abelian gauge potential corresponding to the zero modes of torus. Note that  $a$  and  $\bar{a}$  satisfy the periodicity  $a \rightarrow a + m + n\tau$  and  $\bar{a} \rightarrow \bar{a} + m + n\bar{\tau}$  ( $m, n \in \mathbf{Z}$ ) where  $\mathbf{Z}$  denotes integer.  $m$  and  $n$  correspond to winding numbers of  $\alpha$  and  $\beta$  cycles, respectively.

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<sup>2</sup>In this case, the moduli space of flat connections on a  $G$ -bundle over  $\Sigma$  can be identified with the moduli space of a stable holomorphic  $G^{\mathbf{C}}$ -bundles on  $\Sigma$ . This mathematical fact is known as Narashimhan-Seshadri theorem [20].

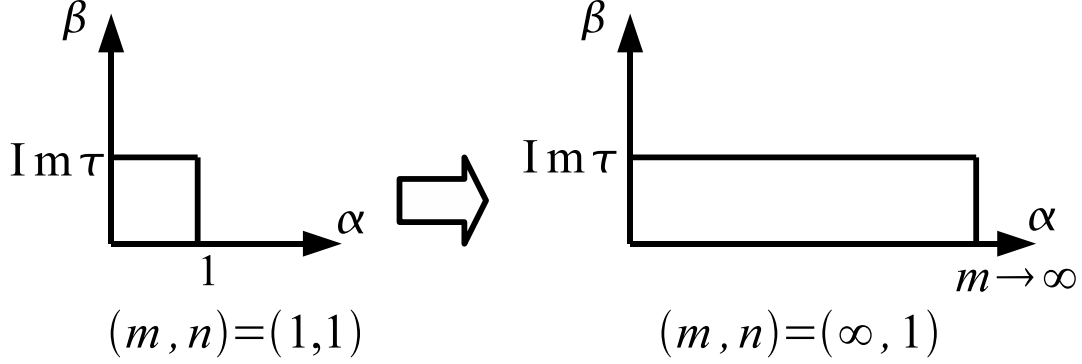


Figure 1: Choice of winding numbers for discussion of deconfinement — we assume  $\text{Re}\tau = 0$  and choose  $mn$  to be a large integer such that  $m \gg n = 1$ . Note that  $\text{Im}\tau$  and  $m$  can be interpreted as radii for the two circles of torus ( $S^1_\alpha \times S^1_\beta$ ) corresponding to  $\alpha$  and  $\beta$  cycles, respectively.

Let us consider a change of variables for the one-forms  $\omega$  and  $\bar{\omega}$  in terms of  $\xi_1$  and  $\xi_2$ . Note that  $\xi_i$  ( $i = 1, 2$ ) take real values in  $0 \leq \xi_i \leq 1$  with  $\xi_i = 0, 1$  being identical. Let  $\omega_1$  and  $\omega_2$  be

$$\begin{aligned}\omega_1 &= (d\bar{z} - dz)/2i = -\text{Im}\tau d\xi_2 \\ \omega_2 &= (\tau d\bar{z} - \bar{\tau} dz)/2i = \text{Im}\tau d\xi_1\end{aligned}\tag{14}$$

where we assume  $\text{Re}\tau = 0$ . Since an integral part of  $\text{Re}\tau$  can be absorbed into  $m$  of the periodicity of  $a \rightarrow a + m + n\tau$ , this assumption is equivalent to  $\text{Re}\tau$  being an integer, which may not cause obstacles in the following discussion.<sup>3</sup> With (14), holonomies of torus can be rewritten as

$$\oint_{\alpha_i} \omega_j = (\text{Im}\tau) \epsilon_{ij}\tag{15}$$

where  $\epsilon_{ij}$  denotes a Levi-Civita symbol and  $\alpha_1, \alpha_2$  denote the the alpha and beta cycles defined in (12). Note that we can set  $\omega(z) = 1$  in (12) with identification of the alpha and beta cycles by loop integrations of the variables  $\xi_1$  and  $\xi_2$ , respectively. Normalization for  $\omega_1$  and  $\omega_2$  is given by

$$\int dz d\bar{z} \frac{\omega_1}{\text{Im}\tau} \wedge \frac{\omega_2}{\text{Im}\tau} = 1.\tag{16}$$

We now introduce a new set of variables corresponding to  $\omega_1, \omega_2$  by

$$a_1 = \bar{a} - a, \quad a_2 = \tau \bar{a} - \bar{\tau} a.\tag{17}$$

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<sup>3</sup>One might argue the assumption of  $\text{Re}\tau = 0$  is relevant to the imaginary time formulation as discussed in the introduction but the relevance is not entirely clear at least for the author.

Under the transformations of  $a \rightarrow a + m + n\tau$  and  $\bar{a} \rightarrow \bar{a} + m + n\bar{\tau}$ ,  $a_1$  and  $a_2$  vary as

$$\begin{aligned}\delta a_1 &\rightarrow (-2i\text{Im}\tau)n, \\ \delta a_2 &\rightarrow (2i\text{Im}\tau)m.\end{aligned}\tag{18}$$

From (15) and (18), we find

$$\exp\left(\oint_{\alpha_2} \frac{\pi\omega_1}{\text{Im}\tau} \frac{\delta a_2}{\text{Im}\tau}\right) = e^{-i2\pi m}, \quad \exp\left(\oint_{\alpha_1} \frac{\pi\omega_2}{\text{Im}\tau} \frac{\delta a_1}{\text{Im}\tau}\right) = e^{-i2\pi n}.\tag{19}$$

For a nonabelian case with an  $SU(N)$  gauge group, physical variables  $a, \bar{a}$  are given by the following matrix-valued quantities [21, 23]:

$$a = a_j t_j^{\text{diag}}, \quad \bar{a} = \bar{a}_j t_j^{\text{diag}}\tag{20}$$

where  $t_j^{\text{diag}}$  are the diagonal generators of  $G = SU(N)$  in the fundamental representation ( $j = 1, 2, \dots, N-1$ ), corresponding to the Cartan subalgebra of  $G$ .  $a_j$  are complex variables satisfying  $a_j \rightarrow a_j + m_j + n_j\tau$  with  $m_j$  and  $n_j$  being integer. In the expressions of (20), sums over  $j$  should be understood.

Nonabelian versions of  $a_1$  and  $a_2$  can also be given by (17) with  $a$  and  $\bar{a}$  now defined as (20). By use of such  $a_1$  and  $a_2$ , we can express matrix parametrization of gauge potentials on torus, which is analogous to the planar case (1), as

$$\begin{aligned}\tilde{A}_{\xi_1} &= A_{\xi_1} + M \left( \frac{\pi\omega_2}{\text{Im}\tau} \frac{a_1}{\text{Im}\tau} \right) M^{-1}, \quad A_{\xi_1} = -\partial_{\xi_1} M M^{-1} \\ \tilde{A}_{\xi_2} &= A_{\xi_2} + M^{\dagger-1} \left( \frac{\pi\omega_1}{\text{Im}\tau} \frac{a_2}{\text{Im}\tau} \right) M^{\dagger}, \quad A_{\xi_2} = M^{\dagger-1} \partial_{\xi_2} M^{\dagger}\end{aligned}\tag{21}$$

where  $\partial_{\xi_i}$  denotes  $\frac{\partial}{\partial \xi_i}$  ( $i = 1, 2$ ). Note that there is a set of  $\epsilon_{ij}\omega a_j$  combinations appeared in (19) such that we have invariance of  $\tilde{A}_{\xi_1}, \tilde{A}_{\xi_2}$  under the transformations of  $(a, \bar{a})$ . In terms of  $a, \bar{a}$ , the parametrization can be expressed as

$$\begin{aligned}\tilde{A}_z &= -\partial_z \tilde{M} \tilde{M}^{-1} \\ &= -\partial_z M M^{-1} + M \left( \frac{\pi\omega}{\text{Im}\tau} \bar{a} \right) M^{-1},\end{aligned}\tag{22}$$

$$\begin{aligned}\tilde{A}_{\bar{z}} &= \tilde{M}^{\dagger-1} \partial_{\bar{z}} \tilde{M}^{\dagger} \\ &= M^{\dagger-1} \partial_{\bar{z}} M^{\dagger} + M^{\dagger-1} \left( \frac{\pi\bar{\omega}}{\text{Im}\tau} a \right) M^{\dagger}\end{aligned}\tag{23}$$

with  $\tilde{M}$  and  $\tilde{M}^{\dagger}$  now defined by

$$\begin{aligned}\tilde{M} &= M \exp\left(-\frac{\pi}{\text{Im}\tau} \int^z \omega \bar{a}\right) \equiv M \tilde{\gamma}_z, \\ \tilde{M}^{\dagger} &= \exp\left(\frac{\pi}{\text{Im}\tau} \int^{\bar{z}} \bar{\omega} a\right) M^{\dagger} \equiv \tilde{\gamma}_{\bar{z}} M^{\dagger}.\end{aligned}\tag{24}$$

Equations (22) and (23) agree with previously known matrix parametrization for gauge potentials on torus [21, 23].



### A gauge invariant measure and decomposition of $\widetilde{H}$

Let us now consider a gauge invariant measure which incorporates the zero modes of torus. From the calculation of the planar case shown in (4) and from Narashimhan-Seshadri theorem, we can define the toric measure as

$$d\mu(\widetilde{\mathcal{C}}) = d\mu(\widetilde{H}) e^{2c_A \mathcal{S}_{WZW}(\widetilde{H})} \quad (25)$$

where  $\widetilde{H} = \widetilde{M}^\dagger \widetilde{M} = \widetilde{\gamma}_z M^\dagger M \widetilde{\gamma}_z = \widetilde{\gamma}_z H \widetilde{\gamma}_z$ . The change from  $H$  to  $\widetilde{H}$  can essentially be taken care of by replacing  $\partial_{\bar{z}}$  with  $\partial_{\bar{z}} + \frac{\pi\bar{\omega}}{\text{Im}\tau}a$  and similar for  $\partial_z$ .

Decomposition of  $a, \bar{a}$  out of  $\widetilde{H}$  may be achieved by imposing

$$[a, H] = a_j [t_j^{\text{diag}}, H] = 0 \quad (a_j \neq 0) \quad (26)$$

This is a strong assumption we would like to impose on  $H$  later in the next section. In terms of matrix configuration, this basically leads to diagonalization of  $H$  for arbitrary choices of  $a_j$ 's ( $j = 1, 2, \dots, N-1$ ). Note that since we are interested in nontrivial zero-mode contributions we consider non-vanishing  $a$  or  $\bar{a}$ , *i.e.*, at least one of  $a_j$ 's should be non-zero. For example, we can choose  $a_l = 0$  ( $l = 1, 2, \dots, r; 1 < r < N-1$ ) and  $a_m \neq 0$  ( $l = r, r+1, \dots, N-1$ ). Then  $H$  does not get fully diagonalized but has a block-diagonal structure with two blocks, one of which being an  $r$ -dimensional block. Under the assumption of (26), we can express the gauge-invariant measure as

$$\begin{aligned} d\mu(\widetilde{\mathcal{C}}) \Big|_{\text{decom.}} &= d\mu(H) d\mu(a, \bar{a}) e^{2c_A \mathcal{S}_{WZW}(\widetilde{H})} \Big|_{[a, H]=0} \\ &= d\mu(\mathcal{C}) d\mu(a, \bar{a}) \Big|_{[a, H]=0} \end{aligned} \quad (27)$$

where  $d\mu(a, \bar{a}) = \prod_{j=1}^{N-1} d\mu(a_j, \bar{a}_j)$ . In the last step, we neglect  $(a, \bar{a})$ -contributions of  $\mathcal{S}_{WZW}(\widetilde{H})$ , which may be absorbed into definitions of wave functions. Since  $d\mu(a, \bar{a})$  is invariant under  $a_j \rightarrow a_j + m_j + n_j \tau$ , the expression (27) shows explicit gauge invariance of the measure which is incorporated with the zero modes.

## 4 Zero-mode Kähler potentials and gauge invariance

In this section, we construct a wave function corresponding to the vacuum state of (2+1)-dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$ .

### Abelian Case

We first focus on abelian zero-mode dynamics and then move to a nonabelian case later. As in the previous section, we shall denote  $a$  as a complex variable for the moment. For an abelian case, we can substitute  $M^\dagger = e^{i\theta(z, \bar{z})}$ , where  $\theta(z, \bar{z})$  is a function of  $z, \bar{z}$ , into (23) to obtain  $\widetilde{A}_{\bar{z}} = i\partial_{\bar{z}}\theta + \frac{\pi\bar{\omega}}{\text{Im}\tau}a$ . Thus, under a certain gauge, physical variables of  $\widetilde{A}_{\bar{z}}$  can be given solely by  $a$  which satisfies the periodicity  $a \rightarrow a + m + n\tau$ .

We now consider geometric quantization of the  $U(1)$  Chern-Simons theory, following the line of [22, 35] in a slightly different manner. From (9), (13) and (22)-(24), we can express the Kähler form of zero modes as

$$\Omega = \frac{k_{a\bar{a}}}{2\pi} da \wedge d\bar{a} \int_{z,\bar{z}} \left( \frac{\pi\bar{\omega}}{\text{Im}\tau} \right) \wedge \left( \frac{\pi\omega}{\text{Im}\tau} \right) = i \frac{\pi k_{a\bar{a}}}{\text{Im}\tau} da \wedge d\bar{a} \quad (28)$$

where the integral is taken over  $dzd\bar{z}$  and  $k_{a\bar{a}}$  is the level number associated to the abelian Chern-Simons theory. The corresponding zero-mode Kähler potentials can generally be expressed as

$$W(a, \bar{a}) = \frac{\pi k_{a\bar{a}}}{\text{Im}\tau} a\bar{a} + g(a) + \bar{g}(\bar{a}) \quad (29)$$

where  $g(a)$  and  $\bar{g}(\bar{a})$  are purely  $a$ -dependent and  $\bar{a}$ -dependent functions, respectively.

From (13)-(17) we have a following relation

$$da \wedge d\bar{a} \int_{z,\bar{z}} \bar{\omega} \wedge \omega = da_1 \wedge da_2 \int_{z,\bar{z}} \frac{\omega_2}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} \quad (30)$$

where we use

$$\omega = \frac{\omega_2 - \tau\omega_1}{\text{Im}\tau}, \quad \bar{\omega} = \frac{\omega_2 - \bar{\tau}\omega_1}{\text{Im}\tau}. \quad (31)$$

In terms of  $a_1$  and  $a_2$ , the zero-mode Kähler form can then be expressed as

$$\begin{aligned} \Omega &= \frac{k_{a\bar{a}}}{2\pi} \left( \frac{\pi}{\text{Im}\tau} \right)^2 da \wedge d\bar{a} \int_{z,\bar{z}} \bar{\omega} \wedge \omega = \frac{k_{a\bar{a}}}{2\pi} \left( \frac{\pi}{\text{Im}\tau} \right)^2 (2i\text{Im}\tau) da \wedge d\bar{a} \\ &= \frac{k_{a\bar{a}}}{2\pi} \left( \frac{\pi}{\text{Im}\tau} \right)^2 da_1 \wedge da_2 \int_{z,\bar{z}} \frac{\omega_2}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} = -\frac{k_{a\bar{a}}}{2\pi} \left( \frac{\pi}{\text{Im}\tau} \right)^2 da_1 \wedge da_2. \end{aligned} \quad (32)$$

A Kähler potential corresponding to the second line in (32) may be given by

$$K(a, \bar{a}) = \frac{i\pi k_{a\bar{a}}}{2(\text{Im}\tau)^2} (\bar{a} - a)(\tau\bar{a} - \bar{\tau}a). \quad (33)$$

$K(a, \bar{a})$  appears to differ from the general expression  $W(a, \bar{a})$  in (29). But this does not cause a problem since both  $W(a, \bar{a})$  and  $K(a, \bar{a})$  are derived from the same Kähler form  $\Omega$  with different choices of frames, *i.e.*, the two Kähler potentials describe the same physics of zero modes. We shall choose  $K(a, \bar{a})$  as our zero-mode Kähler potential in the following.

The symplectic potential for the zero modes can be expressed as

$$\begin{aligned} \mathcal{A} &= \frac{\pi k_{a\bar{a}}}{4(\text{Im}\tau)^2} \int_{z,\bar{z}} \left( \frac{\omega_2 a_1}{\text{Im}\tau} \wedge \frac{\omega_1}{\text{Im}\tau} da_2 - \frac{\omega_1 a_2}{\text{Im}\tau} \wedge \frac{\omega_2}{\text{Im}\tau} da_1 \right) \\ &= -\frac{\pi k_{a\bar{a}}}{4(\text{Im}\tau)^2} (a_1 da_2 + a_2 da_1). \end{aligned} \quad (34)$$

Note that we take account of the couplings of  $a_i$  to  $\omega_i$  ( $i = 1, 2$ ) in  $K(a, \bar{a})$ . A naive calculation of  $d\mathcal{A}$  with respect to  $a_i$  does not lead to  $\Omega$  of (32) but this is not a discrepancy since  $\Omega$  is

also defined with  $a_i$  coupled to  $\omega_i$ . From (18), a variation of  $\mathcal{A}$  under  $a \rightarrow a + m + n\tau$  is given by  $\mathcal{A} \rightarrow \mathcal{A} + d\Lambda_{m,n}$  where

$$\Lambda_{m,n} = -i \frac{\pi k_{a\bar{a}}}{2\text{Im}\tau} (ma_1 - na_2). \quad (35)$$

A holomorphic wavefunction which satisfies the polarization condition can be expressed as

$$\Psi[\tilde{A}_{\bar{z}}] \equiv \Psi[a] = e^{-\frac{K(a,\bar{a})}{2}} f(a). \quad (36)$$

We require zero-mode ‘gauge’ invariance on  $\Psi[a]$  under  $a \rightarrow a + m + n\tau$  by imposing

$$e^{i\Lambda_{m,n}} \Psi[a] = \Psi[a + m + n\tau]. \quad (37)$$

This leads to the following relation

$$f(a) = e^{-i\pi k_{a\bar{a}} mn} f(a + m + n\tau). \quad (38)$$

Since the periodicity property  $f(a) = f(a + m + n\tau)$  is a natural requirement for any functions defined on torus, the relation (38) means that  $\Psi[a]$  can be ‘gauge’ invariant given that  $f(a)$  satisfies a Dirac-like quantization condition for  $k_{a\bar{a}}$ , *i.e.*,  $k_{a\bar{a}} \in 2\mathbf{Z}$ . This is another indication of level quantization for the Chern-Simons theory on torus.

For choices of arbitrary winding numbers  $(m, n)$ , one may make  $n$  be absorbed into  $k_{a\bar{a}}$ . This arrows us to identify  $2n$  with the level number of abelian Chern-Simons theory encoding the zero mode dynamics. We shall later choose  $mn$  to be a large integer such that  $m \gg n = 1$  as mentioned in Fig.1.

An inner product of the holomorphic wavefunctions can be expressed as

$$\langle 1|2 \rangle = \int d\mu(a, \bar{a}) e^{-K(a,\bar{a})} \overline{f_1(a)} f_2(a) \quad (39)$$

where  $\overline{f_1(a)}$  denotes the complex conjugate of the function  $f_1(a)$ . Note that, as a requirement for a holomorphic function, the factor of  $\frac{\pi k_{a\bar{a}}}{\text{Im}\tau} \bar{a}$  is realized by an operation of  $\frac{\partial}{\partial a}$  on  $f(a)$ . We shall discuss this point later below equation (60).

### Nonabelian Case

Let us now turn to the main part of the present paper. From (8)-(10) and Narashimhan-Seshadri theorem, we can express a wave functional for vacuum states of  $(2+1)$ -dimensional Yang-Mills theory on torus as

$$\Psi[\tilde{A}_{\bar{z}}] \equiv \Psi[\tilde{M}^\dagger] = e^{-\frac{\tilde{K}}{2}} e^{\tilde{k} S_{WZW}(\tilde{M}^\dagger)} \Upsilon(a) \quad (40)$$

where  $a$  now has an algebraic structure as in (20) and  $\tilde{K}$  is a toric version of (9), *i.e.*,

$$\tilde{K} = -\frac{\tilde{k}}{\pi} \int_{\Sigma} \text{Tr}(\tilde{A}_{\bar{z}} \tilde{A}_z) \quad (41)$$

with  $\tilde{k}$  being a toric version of the level number  $k$  defined in (9). Note that  $\Upsilon(a)$  in (40) is some functions of  $a$  which does not depend on  $\widetilde{H} = \widetilde{M}^\dagger \widetilde{M}$  as  $\Psi[\widetilde{M}^\dagger]$  being a wave functional for the vacuum states. The inner product can be given by

$$\begin{aligned} \langle 1|2 \rangle &= \int d\mu(\widetilde{C}) \Psi_1^*[\widetilde{M}^\dagger] \Psi_2[\widetilde{M}^\dagger] \\ &= \int d\mu(\widetilde{H}) e^{(2c_A + \tilde{k})\mathcal{S}_{WZW}(\widetilde{H})} \overline{\Upsilon_1(a)} \Upsilon_2(a) \end{aligned} \quad (42)$$

where  $\overline{\Upsilon_1(a)}$  is a complex conjugate of  $\Upsilon_1(a)$  and  $\Psi_1^*[\widetilde{M}^\dagger]$  is defined by

$$\Psi^*[\widetilde{M}^\dagger] \equiv \Psi[\widetilde{M}] = e^{-\frac{\tilde{k}}{2}} e^{\tilde{k}\mathcal{S}_{WZW}(\widetilde{M})} \overline{\Upsilon(a)} \quad (43)$$

We may naively follow the lines of discussion in the planar case to assume  $\tilde{k} \rightarrow 0$  but this implies ignorance of nontrivial zero modes; as we will see in a moment, the level number  $\tilde{k}$  can essentially be given by the zero-mode level number  $k_{a\bar{a}}$  in the previous subsection. To circumvent the problem, we can make a dimensional analysis. One of the interesting features of Yang-Mills theory in three dimensions is that a coupling constant  $e$  has mass dimension of  $\frac{1}{2}$ . Gauge potentials also have  $\frac{1}{2}$  mass dimension for the three-dimensional theory. Conventionally, this is realized by absorption of  $e$  into dimensionless  $A_{\bar{z}}$  in the form of (1). Similarly, we regard  $\text{Im}\tau$  in  $\widetilde{A}_{\bar{z}}$  of (22) as dimensionless. Mass dimension of  $\text{Im}\tau$ , however, can be expressed as  $[(\text{Im}\tau)^{-1}] = [e^2] = 1$ . We can then consider  $\frac{\pi}{\text{Im}\tau}$  as a unit of the level number  $\tilde{k}$ , where the factor of  $\pi$  for  $\text{Im}\tau^{-1}$  arises from periodicity conditions (19) or from the matrix parametrization (22), (23). In terms of dimensionful parameters, we may explicitly write the level number  $\tilde{k}$  as  $\frac{\pi}{\text{Im}\tau e^2} \tilde{k}$ . This vanishes as  $\text{Im}\tau \rightarrow \infty$ , which corresponds to the planar case. For finite  $\text{Im}\tau$  with nonzero  $k_{a\bar{a}}$ , however, the toric level number does not vanish and we have a critical value of  $\frac{1}{\text{Im}\tau}$  at which the factor of  $2c_A + \tilde{k}$  in (42) vanishes. In the KKN Hamiltonian approach, this factor, or the coefficient of  $\mathcal{S}_{WZW}(H)$ , involves in calculations of string tension and mass gap. We can therefore read off a deconfinement temperature from the critical value of  $\frac{1}{\text{Im}\tau}$ , which we shall discuss in the next section.

To understand the relation between  $\tilde{k}$  and  $k_{a\bar{a}}$ , we now consider decomposition of  $a$ ,  $\bar{a}$  out of  $\widetilde{H}$  with an imposition of (26). The Kähler potential (41) can be expressed as

$$\begin{aligned} \widetilde{K}_{decom.} &= -\frac{\tilde{k}}{\pi} \int_{\Sigma} \text{Tr}(A_{\bar{z}} A_z) - \frac{\tilde{k}}{\text{Im}\tau} \int_{\Sigma} \text{Tr}(A_{\bar{z}} M \omega \bar{a} M^{-1} + M^{\dagger-1} \bar{\omega} a M^{\dagger} A_z) \\ &\quad + \prod_j^{N-1} \frac{\pi \tilde{k}}{2\text{Im}\tau} a_j \bar{a}_j \end{aligned} \quad (44)$$

The last term can be written as  $\prod_j \frac{\pi k_{a\bar{a}}}{\text{Im}\tau} a_j \bar{a}_j$  with an identification of

$$\tilde{k} = 2k_{a\bar{a}}. \quad (45)$$

As we have discussed below (38), we can express  $k_{a\bar{a}} = 2n$  for any integer  $n$ . So the identification (45) means  $\tilde{k} = 4n$ . Note that we may choose a sign of  $\tilde{k}$  since the order of variables does not matter in Kähler *potentials* of zero modes; this implies a change of sign

for (positive)  $k_{a\bar{a}}$  in the definition of zero-mode Kähler *form* in (28). As discussed earlier, by a change of frames we may replace  $\frac{\pi k_{a\bar{a}}}{\text{Im}\tau} a_j \bar{a}_j$  with  $K(a_j, \bar{a}_j)$  of (33). Upon decomposition, the wave functional (40) can then be written as

$$\begin{aligned} \Psi[\widetilde{M}^\dagger]_{decom.} &= e^{-\frac{1}{2} \prod_j K(a_j, \bar{a}_j)} \Upsilon(a) e^{\frac{\tilde{k}}{2\pi} \int_\Sigma \text{Tr}(A_{\bar{z}} A_z)} e^{\tilde{k} \mathcal{S}_{WZW}(M^\dagger)} \\ &\quad \times \exp \left[ \frac{\tilde{k}}{2\text{Im}\tau} \int_\Sigma \text{Tr}(A_{\bar{z}} M \omega \bar{a} M^{-1} + M^{\dagger-1} \bar{\omega} a M^\dagger A_z) \right] \\ &\quad \times \exp \left[ -\frac{\tilde{k}}{\text{Im}\tau} \int_\Sigma \text{Tr}(\bar{\omega} a \partial_z M^\dagger M^{\dagger-1}) \right] \end{aligned} \quad (46)$$

where we use the Polyakov-Wiegmann identity (11) for  $\mathcal{S}_{WZW}(\widetilde{M}^\dagger) = \mathcal{S}_{WZW}(\tilde{\gamma}_{\bar{z}} M^\dagger)$  and  $\partial_{\bar{z}} \tilde{\gamma}_{\bar{z}} = \frac{\pi \bar{\omega}}{\text{Im}\tau} a \tilde{\gamma}_{\bar{z}}$ ,  $\partial_z \tilde{\gamma}_{\bar{z}} = 0$  for  $\tilde{\gamma}_{\bar{z}}$  defined in (24). In analogy with the abelian case, we can now regard  $\Upsilon(a)$  as a general function satisfying invariance under  $a_j \rightarrow a_j + m_j + \tau n_j$  for  $a = a_j t_j^{\text{diag}}$  and  $m_j, n_j \in \mathbf{Z}$ . We can have a similar expression for  $\Psi[\widetilde{M}]_{decom.}$  and its product with  $\Psi[\widetilde{M}^\dagger]_{decom.}$  can be written as

$$\begin{aligned} \Psi_1^* \Psi_2]_{decom.} &= e^{-\prod_j K(a_j, \bar{a}_j)} \overline{\Upsilon_1(a)} \Upsilon_2(a) e^{\tilde{k} \mathcal{S}_{WZW}(H)} \\ &\quad \times \exp \left[ \frac{\tilde{k}}{\text{Im}\tau} \int_\Sigma \text{Tr}(H^{-1} \partial_{\bar{z}} H \omega \bar{a} - \bar{\omega} a \partial_z H H^{-1}) \right]. \end{aligned} \quad (47)$$

Earlier we have obtained a gauge invariant wave function for an abelian case but for nonabelian cases it is not possible to do so. This is because a corresponding vacuum wave functional has properties arising from a WZW action. Note that neither of  $\Psi[\widetilde{M}^\dagger]$  or  $\Psi[\widetilde{M}]$  is gauge invariant in terms of the zero mode variables. This is related to the fact that there is no gauge invariant WZW action for  $\mathcal{S}_{WZW}(\widetilde{M}^\dagger) = \mathcal{S}_{WZW}(\tilde{\gamma}_{\bar{z}} M^\dagger)$ . Technically speaking,  $\mathcal{S}_{WZW}(\widetilde{M}^\dagger)$  does not satisfy the so-called anomaly-free condition [26], which is a sort of chirality condition in terms of transformations from  $M^\dagger$  to  $\widetilde{M}^\dagger$ . On the other hand,  $\mathcal{S}_{WZW}(\tilde{H})$  satisfies the anomaly condition. So we may obtain a gauge invariant value for the product in (47). Indeed, we can introduce the following gauged WZW action [24, 25, 26]:

$$\begin{aligned} I(H, a) &= \mathcal{S}_{WZW}(H) + \frac{1}{\pi} \int_\Sigma \text{Tr} \left( H^{-1} \partial_{\bar{z}} H \frac{\pi \omega}{\text{Im}\tau} \bar{a} - \frac{\pi \bar{\omega}}{\text{Im}\tau} a \partial_z H H^{-1} \right. \\ &\quad \left. + H^{-1} \frac{\pi \bar{\omega}}{\text{Im}\tau} a H \frac{\pi \omega}{\text{Im}\tau} \bar{a} - \frac{\pi \bar{\omega}}{\text{Im}\tau} a \frac{\pi \omega}{\text{Im}\tau} \bar{a} \right). \end{aligned} \quad (48)$$

The inner product is then given by

$$\langle 1|2 \rangle_{decom.} = \int d\mu(H) d\mu(a, \bar{a}) e^{(2c_A + \tilde{k}) I(H, a)} \overline{\Psi_1(a)} \Psi_2(a) \Big|_{[a, H]=0} \quad (49)$$

where  $\Psi(a)$  is now defined as

$$\Psi(a) = \exp \left( - \prod_j^{N-1} \frac{K(a_j, \bar{a}_j)}{2} \right) \Upsilon(a). \quad (50)$$

$\overline{\Psi(a)}$  is a complex conjugate of  $\Psi(a)$ .  $\Upsilon(a)$  is a general function on torus with a Cartan subalgebra structure for  $a$ . Equation (49) shows manifest gauge invariance of the inner product in terms of  $H$  and  $(a, \bar{a})$ , including the measure.

What we have done here is to obtain a lower class of the vacuum wave functional (40) and a corresponding inner product by imposition of the decomposition assumption (26). The decomposition condition leads to expressions in terms of  $H$  and  $(a, \bar{a})$  by ‘gauging away’  $(a, \bar{a})$ -dependence in  $\widetilde{H} = \tilde{\gamma}_z H \tilde{\gamma}_z$  so that we can extract the gauge invariant matrix  $H$ . In (47) we find a coupling between the current of  $S_{WZW}(H)$ ,  $\partial_z H H^{-1}$ , and the zero mode variable  $\bar{\omega}a$ . This coupling implies an identification of the current as a non-perturbative gluon field as discussed in the Hamiltonian approach. This is an interesting point, however, the upshot of the decomposition analysis here lies in gauge invariance of the inner product which leads to the relation of level number  $\tilde{k}$  with  $k_{a\bar{a}}$  as in (45) and properties of  $\Upsilon(a)$  as a general non-abelian function on torus. Since  $a = a_j t_j^{\text{diag}}$  is diagonal, the decomposition condition (26) does not affect on the properties of  $\Upsilon(a)$ ; hence, that in (40) remains the same as a Cartan subalgebraic function on torus throughout the present subsection.

The expression (49) suggests that the effects of zero modes on torus can be interpreted as changes in level numbers of the WZW action. In this sense,  $(2+1)$ -dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  may be regarded as Yang-Mills-Chern-Simons theory. This is, however, not in contradiction to our consideration of pure Yang-Mills theory because of the following. As mentioned in the introduction, physical states of  $(2+1)$ -dimensional Yang-Mills theory can be described by holomorphic wave functionals of Chern-Simons theory. The apparent changes of level numbers in the WZW action arise from the zero-mode contributions to the holomorphic wave functionals of Chern-Simons theory in the toric case. Thus we are considering pure Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$ , with the zero-mode contributions represented by  $\tilde{k}$  in the exponent of (49) along with an additional abelian measure  $d\mu(a, \bar{a})$ .

### Vacuum states and theta functions

So far we have chosen a Kähler potential of either abelian or nonabelian theory such that a holomorphic function,  $f(a)$  or  $\Upsilon(a)$ , has a obvious periodic relation characterized by (38). We can however use different Kähler potentials to start with, as long as the potentials lead to the same Kähler form. For example, it is known a certain choice of Kähler potential gives rise to theta functions for holomorphic functions for an abelian case [22, 35]. We shall briefly review this fact in the rest of this section for better understanding of the above mentioned results in the framework of geometric quantization.

A Kähler potential we choose is

$$W_\tau(a, \bar{a}) = \frac{i\pi k_{a\bar{a}}}{2(\text{Im}\tau)^2} (\bar{a} - a)^2 \tau \quad (51)$$

which, along with an implicit assumption of  $\text{Re}\tau = 0$ , can be expressed in the form of (29) and leads to the Kähler form (28). A polarization condition for a holomorphic function  $\Psi_\tau(a)$  is given by  $\mathcal{D}_{\bar{a}}\Psi_\tau(a) = (\partial_{\bar{a}} - i\mathcal{A}_{\bar{a}})\Psi_\tau(a) = 0$  where  $\mathcal{A}_{\bar{a}} = \frac{i}{2}\partial_{\bar{a}}W_\tau(a, \bar{a})$ . The holomorphic

function is then given by

$$\Psi_\tau(a) = \exp \left[ \frac{-i\pi k_{a\bar{a}}}{4(\text{Im}\tau)^2} (\bar{a} - a)^2 \tau \right] f_\tau(a). \quad (52)$$

This is a general expression for a holomorphic functions upon a choice of the Kähler potential as we have seen earlier. Properties of a holomorphic function  $f_\tau(a)$  can similarly be obtained by imposing gauge invariance on  $\Psi_\tau(a)$ . With a choice of the corresponding symplectic one-form of the form [22, 23]:

$$\mathcal{A}_\tau = - \frac{\pi k_{a\bar{a}}}{2(\text{Im}\tau)^2} (\bar{a} - a)(\tau d\bar{a} - \bar{\tau} da), \quad (53)$$

we have  $\delta\mathcal{A}_\tau = d\Lambda_{\tau m, n}$  for transformations of  $a \rightarrow a + m + n\tau$  with

$$\Lambda_{\tau m, n} = \frac{i\pi k_{a\bar{a}}}{\text{Im}\tau} n(\tau\bar{a} - \bar{\tau}a). \quad (54)$$

Gauge invariance on  $\Psi_\tau(a)$  is then given by  $e^{i\Lambda_{\tau m, n}} \Psi_\tau(a) = \Psi_\tau(a + m + n\tau)$ . This leads to the following relation

$$f_\tau(a) = e^{i2\pi k_{a\bar{a}} \left( \frac{n^2}{2} \tau + an \right)} f_\tau(a + m + n\tau). \quad (55)$$

This shows that  $f_\tau(a)$  is a Jacobi  $\theta$ -function defined by

$$\theta(a, \tau) = \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (a, \tau) \quad (56)$$

where

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{n \in \mathbf{Z}} e^{i\pi k_{a\bar{a}} \tau (n+a)^2 + 2i\pi k_{a\bar{a}} (n+a)(z+b)}. \quad (57)$$

An operation of  $\frac{\partial}{\partial a}$  on  $f_\tau(a)$  corresponds  $\frac{\pi k_{a\bar{a}}}{\text{Im}\tau} (\bar{a} - a)$ . acting on  $f_\tau(a)$ . This is in consistent with the Kähler form written by  $\Omega = -i \frac{\pi k_{a\bar{a}}}{\text{Im}\tau} d(\bar{a} - a) \wedge da$ . In terms of  $f_\tau(a)$ , the inner product for the holomorphic functions is expressed by

$$\langle 1|2 \rangle = \int d\mu(a, \bar{a}) e^{-W_\tau(a, \bar{a})} \overline{f_{\tau 1}(a)} f_{\tau 2}(a). \quad (58)$$

Expanding the Kähler potential, we can rewrite this as

$$\langle 1|2 \rangle = \int d\mu(a, \bar{a}) e^{-\frac{\pi k_{a\bar{a}}}{\text{Im}\tau} a\bar{a}} \overline{g_{\tau 1}(a)} g_{\tau 2}(a) \quad (59)$$

where we introduce

$$g_\tau(a) = \exp \left[ \frac{\pi k_{a\bar{a}} a^2}{2\text{Im}\tau} \right] f_\tau(a). \quad (60)$$

We now find the operation of  $\frac{\partial}{\partial a}$  on  $g_\tau(a)$  is realized by  $\frac{\pi k_{a\bar{a}}}{\text{Im}\tau} \bar{a}$ . We further find that

$$g_\tau(a) = g_\tau(a + m + n\tau) \quad (m = 0, \text{Re}\tau = 0) \quad (61)$$

regardless a choice of  $k_{a\bar{a}}$ . This periodic relation is a subsector of the relation for more general holomorphic function  $f(a)$  in (38) since relation (61) is realized when  $m = 0$  and  $\text{Re}\tau = 0$  are satisfied. The choice of  $f(a) = g_\tau(a)$  is then a concrete realization of the relation

$$\frac{\partial}{\partial a} f(a) = \frac{\pi k_{a\bar{a}}}{\text{Im}\tau} \bar{a} f(a). \quad (62)$$

What is essential in construction of wave functions of zero modes and a corresponding inner product is the Kähler form to start with. Different Kähler potentials may lead to different expressions, *e.g.*, (58) and (59), yet physical consequences should be unaltered. With such a principle, we may require the relation (62) for the previously discussed holomorphic function  $f(a)$  in (36).

Extension to a nonabelian case is straightforward with a knowledge of  $SU(N)$  algebra. The theta function is to be replaced by a higher dimensional theta function, more precisely, the Weyl-Kac character for  $SU(N)$  algebra with level number  $k_{a\bar{a}}$  [36]:

$$ch_{\hat{\lambda}}(a, \tau) = \text{Tr}_{\hat{\lambda}} e^{\pi i k_{a\bar{a}} \tau h^2 - 2\pi i k_{a\bar{a}} (a_1 h_1 + a_2 h_2 + \dots + a_{N-1} h_{N-1})} \quad (63)$$

where  $h = h_1 + h_2 + \dots + h_{N-1}$  and the trace means

$$\text{Tr}_{\hat{\lambda}} = \sum_{h \in \mathbf{Z} + \frac{\lambda}{k_{a\bar{a}}}}. \quad (64)$$

Note that for vacuum wave functions, we should take the ground state for  $\hat{\lambda}$ , *i.e.*,  $\lambda = 0$ . A nonabelian version of a vacuum wave functional  $\Psi_0[\tilde{A}_{\bar{z}}]$  can be constructed by use of (63). Such a wave functional has been studied before and is explicitly given in [23].

## 5 Deconfining limit

In this section, we return to a physical part of the present paper. We consider contributions of the zero modes to the planar case by taking the winding numbers in the limit of  $(m, n) = (\infty, 1)$ . Since the zero-mode contribution is described by Chern-Simons theory on torus with level number  $\tilde{k}$ , the effect of zero modes can be evaluated with a replacement of  $2c_A$  with  $(2c_A + \tilde{k})$  as seen in equation (42) (see also [30], for rigorous discussion). From earlier discussion below equation (43), we can express  $\tilde{k}$  as  $-\frac{2\pi k_{a\bar{a}}}{\text{Im}\tau e^2}$  with  $k_{a\bar{a}} = 2n$ . The critical temperature is therefore given by  $\left(\frac{1}{\text{Im}\tau}\right)_c = \frac{e^2 N}{2\pi}$  where we use  $c_A = N$  and  $n = 1$ . The deconfinement temperature is then expressed as

$$T_c = \frac{e^2 N}{2\pi} \quad (65)$$

which is the same as a mass for non-perturbative gluons predicted in the KKN Hamiltonian approach. Thus  $T_c$  in (65) is a natural result and it is what we seek for in the present study. In what follows, we shall briefly review the calculation of string tension in the Hamiltonian approach for completion of our discussion.



The vacuum expectation value of the Wilson loop operator  $\langle W(C) \rangle_0 = \langle \Psi_0 | W(C) | \Psi_0 \rangle$  can be calculated as

$$\langle W(C) \rangle_0 = \int d\mu(\tilde{H}) e^{(2c_A + \tilde{k})S_{WZW}(\tilde{H})} e^{-S(\tilde{H})} \overline{\Upsilon(a)} \Upsilon(a) W(C) \quad (66)$$

where the Wilson loop operator is given by  $W(C) = \text{TrP} \exp(-\oint \tilde{A}) = \text{TrP} \exp\left(\frac{\pi}{c_A} \oint \tilde{J}\right)$  with  $\tilde{J} = \frac{c_A}{\pi} \partial_z \tilde{H} \tilde{H}^{-1}$ . The function  $S(\tilde{H})$  denotes a contribution from the potential energy of the Yang-Mills theory. For modes of low momenta, or for a (continuum) strong coupling limit, this function can be evaluated. Using an analog of two-dimensional Yang-Mills theory and setting  $\Upsilon(a) = 1$ , we can evaluate the vacuum expectation as

$$\langle W(C) \rangle_0 \approx \exp[-\tilde{\sigma} \mathcal{A}_C] \quad (67)$$

where  $\mathcal{A}_C$  is the area of the loop  $C$  and  $\tilde{\sigma}$  is the string tension on torus given by

$$\tilde{\sigma} = \frac{e^4}{4\pi} \left( c_A + \frac{1}{2} \tilde{k} \right) c_F. \quad (68)$$

Here  $c_F = (N-1)(N+1)/2N$  is the quadratic Casimir for  $SU(N)$  in the fundamental representation. Substituting  $\tilde{k} = -\frac{2\pi k_{a\bar{a}}}{\text{Im}\tau e^2}$  ( $k_{a\bar{a}} = 2n$ ,  $n = 1$ ), we find vanishing of the string tension at  $T_c$ .

Temperatures corresponding to  $n > 1$  are irrelevant in our setting. However, we can in fact choose a finite  $n \neq 1$  as long as  $m \gg n \geq 0$  is satisfied and in this case  $\text{Im}\tau$  is scaled to  $n\text{Im}\tau$  such that  $\tilde{k}$  remains the same for any  $n$ . Notice that for the case of  $n = 0$ , which may be possible as we consider  $n$  as the winding number of the beta cycle of torus, we have vanishing of  $n\text{Im}\tau$  but yet  $\tilde{k}$  remains as  $\tilde{k} = -\frac{4\pi}{\text{Im}\tau e^2}$ . In terms of the picture in Fig.1, the choice of  $n = 0$  means that the torus of our interest is dimensionally reduced to a circle (times a point). Thus, in our setting the choice of  $n = 0$  may be ruled out for a dimensional reasoning.

In the planer case, the string tension  $\sigma$  is given by

$$\sigma = e^4 \left( \frac{N^2 - 1}{8\pi} \right). \quad (69)$$

Comparisons with numerical data can be made for dimensionless parameter  $T_c/\sqrt{\sigma}$ . Our prediction for this value is

$$\begin{aligned} \frac{T_c}{\sqrt{\sigma}} &= \sqrt{\frac{2}{\pi}} \sqrt{\frac{N^2}{N^2 - 1}} \\ &= 0.798 \sqrt{\frac{N^2}{N^2 - 1}}. \end{aligned} \quad (70)$$

Lattice simulations show 0.865, 0.903 and 0.86(7) for this value at  $N \rightarrow \infty$ . These values are taken from references [14], [15] and [17], respectively. Corresponding error percentages are 8.40%, 13.2% and 8.65%. Among the lattice data, the one given by [15] actually provides the most updated and reliable result. Obviously, we need to make further investigation to figure out the relatively large deviation between the lattice data and the value (70).

## 6 Concluding remarks

In the present paper, we consider  $(2 + 1)$ -dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  in the framework of the so-called Karabali-Kim-Nair (KKN) Hamiltonian approach. A physical motivation to consider the toric theory is clear since we may regard it as the planar theory at a finite temperature in the limit of a large radius for one of the  $S^1$ 's of torus ( $S^1 \times S^1$ ), and, hence, we can discuss deconfinement transition in terms of the other radius. In order to execute a calculation of a deconfinement temperature, however, we need to understand some mathematical aspects of the KKN Hamiltonian approach. In section 2, we review few features of the Hamiltonian approach which are pertinent to our discussion. Detailed analysis on dynamics or geometry of zero modes of torus is given in section 3 for both abelian and nonabelian cases. For a nonabelian case, we construct vacuum-state wave functionals for  $(2 + 1)$ -dimensional Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  by use of Narashimhan-Seshadri theorem. We further consider a subsector of the vacuum wave functionals by imposing a certain condition (26) to discuss gauge invariance of an inner product of the vacuum wave functionals. Along the way, we also find zero-mode contributions to the planar Yang-Mills theory. In section 4, we compute a string tension of pure Yang-Mills theory on  $S^1 \times S^1 \times \mathbf{R}$  in the Hamiltonian framework, namely, in the so-called continuous strong coupling limit, and find a deconfinement temperature (65). This value agrees with numerical data from lattice simulation quite roughly in 10%. We shall leave the explanation of this rather large error for future studies.

Now we would like to comment on subtle points in the arguments which we have used in the present paper. We argue that we take a cylindrical limit of the torus, *i.e.*,  $S^1 \times S^1 \times \mathbf{R} \rightarrow S^1 \times \mathbf{R}^2$ , in the end of calculations. In our framework, one of the  $S^1$  directions of the torus corresponds to the time coordinate. The Lorentz invariance however implies that we can interchange this temporal direction with the spatial direction  $\mathbf{R}$ . Thus our results suggest that a change of topologies (from plane to cylinder) leads to an apparent change of level numbers in a WZW model which is relevant to the gauge invariant measure in  $(2 + 1)$ -dimensional Yang-Mills theory. This is a nontrivial result and probably contains some subtleties because that a topology change causes a change of level numbers is simply counter-intuitive. In this paper, we present one of the justifications of this issue by use of gauge invariance. We have made the following argument.

In the KKN Hamiltonian approach, the Gauss law constraint (or the integrability) of the gauge potentials should be satisfied regardless what Riemann surfaces we use in construction of the WZW action which is relevant to the gauge invariant measure of  $(2 + 1)$ -dimensional Yang-Mills theory. Thus, the wave functionals in the toric theory can be written as (40). (This is why the use of Narashimhan-Seshadri theorem has been emphasized in the present paper.) The symplectic structure of the zero modes is essentially given by (32). This is not exactly the symplectic structure of Chern-Simons theory. However, with an algebraic extension (20), we can relate the level number  $k_{a\bar{a}}$  of (32) to the level number  $\tilde{k}$  of (40) so that we can encode the zero-mode contributions in the level number  $\tilde{k}$ . A detailed explanation of this relation is given by the argument of gauge invariance in section 4. The argument is limited to a particular case, where we can explicitly write down the gauge invariant measure

in terms of zero modes. Although this will be sufficient to show the relation for our purposes, it is desirable to confirm this relation in more general cases. We shall leave this task for future studies.

Lastly, we would like to emphasize that the validity of our analyses and results is limited in the framework of the KKN Hamiltonian approach. It is within this framework that we can properly use the abelian Chern-Simons symplectic form to discuss zero-mode contributions to  $(2 + 1)$ -dimensional Yang-Mills theory. We have not proven the use of the Chern-Simons symplectic form in general. Furthermore there may be some subtleties to justify this analysis in the more standard approaches to Yang-Mills theory. Although this might be the case, what is significant in the present paper from a physical perspective is that the use of the Chern-Simons symplectic form does seem to lead a reasonable estimate for the deconfinement temperature and that this fact itself suggests the usefulness of the KKN Hamiltonian approach in the future investigations of  $(2 + 1)$ -dimensional Yang-Mills theory.

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